ON A CERTAIN CLASS OF CANONICAL FORMS.*

BY MR. RALPH A. ROBERTS.

AN interesting class of theorems occurs occasionally in the consideration of algebraical quantics, viz., when a quantic (or quantics) is not in general reducible to a form (or forms) which at first sight, when we count the number of constants involved, appears to be sufficiently general to admit of a finite number of reductions. Such cases bear an analogy to the porism in geometry, as the reductions are impossible, except when the quantic (or quantics) satisfies an invariant relation, and then the number of reductions is infinite. These cases are not common in binary quantics, and are not very remarkable when they do occur. An instance is: Let x_1, x_2, x_3 be three linear expressions in the variables. Then $x_1x_2x_3$ and $ax_1^3 + bx_2^3 + cx_3^3 + dx_1x_2x_3$ contain six constants, viz., three involved in x_1, x_2, x_3 and the three external constants $a/b/c/d$; but two general binary cubics which contain six constants cannot be simultaneously reduced to these forms, unless the combinantive invariant obtained by substituting differential symbols in one and operating on the other vanishes. This, however, is readily apparent from the fact that there is an identical linear relation connecting the three cubes x_1^3, x_2^3, x_3^3 and the product $x_1x_2x_3$, so that the second form is less general than it appears to be at first sight.

But in ternary and quaternary quantics there are several striking cases. The most remarkable, perhaps, is that discovered by Lüroth, viz, that a general plane quartic curve cannot be expressed linearly in terms of the fourth powers of five lines; say

$$a_1x_1^4 + a_2x_2^4 + a_3x_3^4 + a_4x_4^4 + a_5x_5^4. \qquad (1)$$

I insert here a proof of this result, as it involves a method which I propose to use farther on.

Consider the unique conic Σ which can be described to touch the five lines. Then substituting differential symbols, i.e., $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, for λ, μ, ν in the tangential equation $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$ of Σ, and operating with the resulting expression on (1), the remainder vanishes identically. For, operating with $(A, B, C, F, G, H)\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^2$ on $x_1^4 = (\alpha_1x + \beta_1y + \gamma_1z)^4$, we get a result proportional to $(A\alpha_1^2 + B\beta_1^2 + C\gamma_1^2 + 2F\beta_1\gamma_1 + 2G\gamma_1\alpha_1 + 2H\alpha_1\beta_1)x_1^2$; and this vanishes because x_1 touches Σ.

* Read before the American Mathematical Society, December 28, 1894.

But with the quartic in a general form, if we operate with the result of substituting differential symbols in the tangential equation of a conic we obtain another conic in its direct form, and in order that this should vanish identically the six coefficients must separately be equal to zero. Thus we can eliminate the six constants in Σ and obtain an invariant relation. The latter is in fact, the determinant formed by the six second differential coefficients of the quartic. (See Salmon's *Higher Plane Curves.*)

Another form mentioned by Salmon is

$$A_1x_2x_3x_4x_5 + \text{etc.} + A_5x_1x_2x_3x_4, \qquad (2)$$

to which a general quartic cannot be reduced.

In this case the proof follows from the fact that this form is a covariant of (1), and thus cannot contain any more constants than the latter does. But another proof can be found by a method given by Darboux in his work "Sur une classe remarquable de courbes et de surfaces algébriques" (Paris, 1873), viz., by using as co-ordinates the parameters of the two tangents which can be drawn from a point to the conic touching the five lines. It will then be seen that the form (2) is one of an infinite number of the same kind.

It may be worth while observing that two cubic curves cannot be written in the forms

$$\left.\begin{aligned} &\alpha_1x_1^3 + \alpha_2x_2^3 + \alpha_3x_3^3 + \alpha_4x_4^3 + \alpha_5x_5^3, \\ &\beta_1x_1^3 + \beta_2x_2^3 + \beta_3x_3^3 + \beta_4x_4^3 + \beta_5x_5^3, \end{aligned}\right\} \qquad (3)$$

although they appear to contain 8 constants externally and 10 constants in the lines; that is, 18 constants altogether, which should be exactly right for two cubic curves with 9 constants each. The proof easily follows as before, and the determinant formed by the six first differential coefficients is found to vanish.

Coming now to three dimensions, I observe that a remarkable case is given in Salmon's *Surfaces.* It is shown there that three quadrics cannot in general be expressed linearly in terms of the squares of five planes, although it would appear that there were a sufficient number of constants; viz., 12 constants externally and 15 implicitly in the five planes, that is, 27 altogether, which should be sufficient for three quadrics with nine constants each. It would also appear that a quaternary quartic could not be expressed as the sum of nine fourth powers, although such a form would appear to contain 35 constants—more than sufficient to express a general quaternary quartic, which contains 34 constants. The proof can be obtained by considering the unique quadric touching the nine

planes; and the determinant formed by the ten second differential coefficients is found to vanish. I am not aware whether this result, as well as that relating to the two plane cubics, has been noticed before.

I now proceed to consider a result I have arrived at of the kind explained above; but before doing so I propose to consider a reduction which leads up to my theorem, as it throws light upon the processes employed in the more general case.

To reduce a conic and a plane cubic curve to the forms

$$\left.\begin{aligned} &\alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_3 x_3^3 + \alpha_4 x_4^3, \\ &a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2. \end{aligned}\right\} \qquad (4)$$

Counting the constants, we have 6 externally and 8 implicitly; that is, 14 altogether, which are exactly right for a cubic and a conic, with nine and five constants, respectively.

Now consider a conic Σ touching the four lines x_1, x_2, x_3, x_4; then if we substitute differential symbols in the tangential equation of Σ and operate on the cubic and the conic, both the results will vanish identically, exactly in the way I have explained before; for the result of the operation on the power of any line, L^n say, is proportional to L^{n-2} multiplied by the condition that L should touch the conic.

But suppose we write the cubic in one of its canonical forms, i.e.,

$$x^3 + y^3 + z^3 + 6mxyz = 0;$$

and that the conic at the same time is $(a, b, c, f, g, h)(x, y, z)^2 = 0$: then, operating with the result of substituting differential symbols in the tangential equation of the other conic Σ $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$ which touches the four lines x_1, x_2, x_3, x_4 in (4), we get

$$(A + 2mF)x + (B + 2mG)y + (C + 2mH)z,$$

and $$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh;$$

and in order that these expressions should vanish identically, we must have, from the first,

$$A + 2mFe = 0, \quad B + 2mG = 0, \quad C + 2mH = 0;$$

thus Σ must be

$$A(m\lambda^2 - \mu\nu) + B(m\mu^2 - \nu\lambda) + C(m\nu^2 - \lambda\mu) = 0,$$

subject to the condition

$$A(ma - f) + B(mb - g) + C(mc - h) = 0;$$

that is, Σ must touch the four lines determined by the equations

$$\frac{m\lambda^2 - \mu\nu}{ma - f} = \frac{m\mu^2 - \nu\lambda}{mb - g} = \frac{m\nu^2 - \lambda\mu}{mc - h}.$$

Hence it is plain that these four lines are the lines x_1, x_2, x_3, x_4 in (4), and that the reduction is unique, as it has appeared that there is only one system of such lines. If it is required to find the constants α_1, etc., a_1, etc., in the forms (4), there is no difficulty in finding equations to determine these quantities.

Now this result in a plane suggests an extension to three dimensions. Suppose we wish to reduce a quaternary cubic and a quadric to the forms

$$\left.\begin{array}{l} \alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_3 x_3^3 + \alpha_4 x_4^3 + \alpha_5 x_5^3 + \alpha_6 x_6^3, \\ a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2 + a_6 x_6^2, \end{array}\right\} \quad (5)$$

respectively. Counting the constants, we have 10 externally and 18 implicitly in the 6 planes; that is, 28 altogether, which are exactly right for a quaternary cubic with 19 constants and a quadric with 9 constants. In proceeding to consider the reduction in this case, the extension is from a conic in the plane to a twisted cubic curve in space. Let us consider a twisted cubic osculating the six planes in (5). This is a definite unique curve, for the problem is precisely the same, though in the tangential sense, as to describe a twisted cubic passing through six points; and the latter admits of a single solution (see Salmon's *Surfaces*). Now (see *loc. cit.*) there are three quadric surfaces touching all the osculating planes of the twisted cubic, which are not connected by a linear relation, precisely as there are three quadric surfaces, not connected by a linear relation, which can be described to pass through the curve. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be the tangential equations of these three quadrics: then from the forms (5) we see that the results of substituting differential symbols $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{du}\right)$ for the tangential co-ordinates in any one of the quadrics, or the quadric $l\Sigma_1 + m\Sigma_2 + n\Sigma_3$, and operating on the cubic surface and quadric vanish identically; for the result of substituting differential symbols in the tangential equation of a quadric and operating on the power of a plane, L^n say, is proportional to L^{n-2} multiplied by the condition that L should touch the quadric.

Now, if we consider the cubic surface in the general or any other form, the result of substituting differential symbols in the tangential equation of a quadric and operating on the

cubic will be of the form $Ax + By + Cz + Du$, and in order that this should vanish identically we should have $A=0$, $B=0$, $C=0$, $D=0$; that is, four conditions. Consequently, from the three non-linearly connected quadrics touching the osculating planes of the twisted cubic curve, we should have, apparently, twelve conditions connecting the curve with the surface, that is, sufficient conditions to determine the curve completely if the surface were given. Then since the curve also should satisfy three conditions with the quadric, it might at a rough inspection appear that three invariant conditions should connect the quadric with the cubic. Such, however, is not the case. The apparent twelve conditions satisfied by the twisted cubic curve with the cubic surface are in reality only equivalent to ten, as I proceed to show.

Let the lines of reference be chosen so that the cubic curve may be represented tangentially by the equations

$$\beta^2 - \gamma\alpha = 0, \quad \alpha\delta - \beta\gamma = 0, \quad \gamma^2 - \beta\delta = 0, \qquad (6)$$

and let the canonical planes of the cubic surface, according to Sylvester's unique canonical form, be

$$\alpha_i x + \beta_i y + \gamma_i z + \delta_i u = 0,$$

so that the cubic surface itself is

$$\Sigma_i^1 (\alpha_i x + \beta_i y + \gamma_i z + \delta_i u)^3 = 0.$$

Then, U being one of the three expressions in (6), if we operate with the result of substituting differential symbols in U on the cubic surface, and express that the result vanishes identically, we get

$$\Sigma\alpha_i U_i = 0, \quad \Sigma\beta_i U_i = 0, \quad \Sigma\gamma_i U_i = 0, \quad \Sigma\delta_i U_i = 0, \quad (7)$$

which, on account of the three values of U, apparently gives twelve conditions. This is, however, not the case, in consequence of the two identical relations,

$$\left.\begin{aligned} \delta(\beta^2 - \gamma\alpha) + \gamma(\alpha\delta - \beta\gamma) + \beta(\gamma^2 - \beta\delta) = 0; \\ \gamma(\beta^2 - \gamma\alpha) + \beta(\alpha\delta - \beta\gamma) + \alpha(\gamma^2 - \beta\delta) = 0. \end{aligned}\right\} \quad (8)$$

Hence there remain ten conditions; and when these are satisfied I propose to show that the five canonical planes are osculating planes of the twisted cubic. Solving the equations (7), we have

$$\frac{U_1}{\Delta_1} = \frac{U_2}{\Delta_2} = \text{etc.} = \frac{U_5}{\Delta_5},$$

where Δ_1, Δ_2, etc., are the determinants formed by the coefficients of the five planes. Hence the ratios U_1/U_2, etc., must have the same values, no matter which of the quadrics U we select; that is,

$$\frac{\beta_1^2 - \gamma_1\alpha_1}{\beta_2^2 - \gamma_2\alpha_2} = \frac{\alpha_1\delta_1 - \beta_1\gamma_1}{\alpha_2\delta_2 - \beta_2\gamma_2} = \frac{\gamma_1^2 - \beta_1\delta_1}{\gamma_2^2 - \beta_2\delta_2}, \text{ etc.}$$

Hence
$$\frac{\beta_1^2 - \gamma_1\alpha_1}{\alpha_1\delta_1 - \beta_1\gamma_1} = \frac{\beta_2^2 - \gamma_2\alpha_2}{\alpha_2\delta_2 - \beta_2\gamma_2}, \text{ etc.}$$

From which it follows that all the five planes must satisfy equations of the form

$$\beta^2 - \gamma\alpha - k(\alpha\delta - \beta\gamma) = 0, \quad \gamma^2 - \beta\delta - k'(\alpha\delta - \beta\gamma) = 0,$$

where k, k' are constants. Hence from the identities (8), if $\beta^2 - \gamma\alpha$, $\alpha\delta - \beta\gamma$ and $\gamma^2 - \beta\delta$ do not all vanish, we should have

$$k\delta + \gamma + k'\delta = 0;$$
$$k\gamma + \beta + k'\alpha = 0.$$

But if such equations were satisfied by the planes, they should all have a line in common, which result is contrary to hypothesis; therefore $\beta^2 - \gamma\alpha$, $\alpha\delta - \beta\gamma$, and $\gamma^2 - \beta\delta$ must all vanish for each of the planes; that is, the planes must be osculating planes of the twisted cubic. Hence the latter curve having five given osculating planes satisfies ten conditions, and therefore has still two degrees of freedom. But operating with the results of substituting differential symbols in the three tangential quadrics on the given quadric we get three conditions, which are one greater than the degrees of freedom. Thus it would appear that one relation must exist between the cubic surface and the quadric. In other words, the cubic surface and the quadric cannot be reduced to the forms (5), unless a certain invariant relation between them is satisfied. I proceed to find this relation in the case when the cubic surface and quadric are written in the forms

$$\left.\begin{array}{l} Ax^3 + By^3 + Cz^3 + Du^3 + Ev^3 = 0, \\ (a, b, c, d, f, g, h, l, m, n)(x, y, z, u)^2 = 0, \end{array}\right\} \qquad (9)$$

where $x + y + z + u + v = 0$, identically.

Now it can easily be verified that a twisted cubic curve osculated by the five planes, x, y, z, u, v, can be represented parametrically thus:

$$\lambda : \mu : \nu : \rho = (t - \alpha)^{-1} : (t - \beta)^{-1} : (t - \gamma)^{-1} : (t - \delta)^{-1},$$

where λ, μ, ν, ρ are tangential co-ordinates. The ratios of the differences of $\alpha, \beta, \gamma, \delta$ give the two degrees of freedom which the curve still possesses. Now by eliminating t we may write down

$$\left.\begin{aligned}(\beta-\gamma)\mu\nu+(\gamma-\alpha)\nu\lambda+(\alpha-\beta)\lambda\mu=0,\\(\beta-\delta)\mu\rho+(\delta-\alpha)\rho\lambda+(\alpha-\beta)\lambda\mu=0,\\(\gamma-\delta)\nu\rho+(\delta-\beta)\rho\mu+(\beta-\gamma)\mu\nu=0,\end{aligned}\right\} \qquad (10)$$

which are three tangential quadrics touching the osculating planes of the curve and not connected by a linear relation. Substituting, then, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{du}$ for λ, μ, ν, ρ, respectively, in these three expressions and operating on the quadric given above, we get

$$\begin{aligned}(\beta-\gamma)f+(\gamma-\alpha)g+(\alpha-\beta)h=0,\\(\beta-\delta)m+(\delta-\alpha)l+(\alpha-\beta)h=0,\\(\gamma-\delta)n+(\delta-\beta)m+(\beta-\gamma)f=0,\end{aligned}$$

whence

$$\begin{aligned}(h-l)(g-f)(f-m)+(h-g)(m-h)(n-f)\\+(h-f)(h-l)(n-f)=0,\end{aligned} \qquad (11)$$

which is consequently the invariant relation connecting the cubic surface and the quadric when they are capable of being written in the forms (5). Thus it appears that a cubic surface and a quadric cannot, in general, be reduced to the forms (5), and that when they are reducible to these forms, the reduction can take place in a singly infinite number of ways, all the planes x_1, x_2, etc., involved being osculating planes of a given twisted cubic.

HAYWARD'S VECTOR ALGEBRA.

The Algebra of Coplanar Vectors and Trigonometry. By R. Baldwin Hayward. Macmillan & Co., 1892. 8vo, pp. xxix + 343.

It is a curious fact that while the English are the one nation which in elementary geometry clings to Euclid, the prototype of mathematical rigor, not only is most recent English mathematical work, however excellent in many respects, decidedly lacking in rigor of form, but many English mathematical writers of the present day show an entire lack of critical sense which if shown in elementary geometry would discredit a schoolboy.

The book we now have under consideration is an attempt to build up the theory of complex quantities and of the simpler functions of a complex variable in a systematic manner, presupposing only a moderate acquaintance with arithmetic and the elementary algebra of real numbers. In a work of this sort it is surely not pedantic to expect a certain amount of logical rigor. This need not perhaps be of the highest sort in which, as in the geometrical system of Euclid or the arithmetical system of Weierstrass and others, every proposition is an immediate deduction from one or more fundamental statements (definitions, axioms, etc.). Each proposition should, however, be a logical deduction from *something* which if not one of the formally stated axioms or definitions, must appear self-evident upon careful consideration.* Nothing of this sort is to be found in this book. Take, for instance, the matter of the "permanence of equivalent forms." This principle is stated, using the words of Peacock, as follows (page 4): "All results of algebra which are general in form, though they have been established as true only for restricted meanings of the symbols, must also be true when the symbols are general in value as well as in form." The language here used is so vague that no definite meaning can be attached to it. Moreover, when this principle is applied to the subject of infinite series it is modified by the requirement that the series in question should converge, but not otherwise. We may mention, for example, the treatment of the binomial theorem for negative or fractional exponents. The author never doubts that whenever we get a convergent series by applying the formula obtained for positive integral exponents this series will converge to the desired value. It will be seen that Mr. Hayward has either never heard of the classic researches of Abel (*Crelle*, vol. I., 1826), or regards them as utterly unnecessary. This one example may suffice. The language is constantly open to misinterpretation, and often, as has just been pointed out, the trouble is not merely in the form of expression.

It will be seen from what has already been said that the book is not one which can safely be placed in the hands of a beginner. To the more mature mathematical reader, however, who is able to sift the grain from the chaff it presents much which is of interest.

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* By this is meant that we may in ordinary mathematical work make use of propositions which we cannot prove, but which our geometrical intuition tells us must be true; as, for instance, the proposition that a continuous curve which crosses a horizontal line twice must have a horizontal tangent at some intermediate point. Even such a slight departure as this from true mathematical rigor may, however, in some extreme cases lead us into error.

There are two ways in which the idea of number can be developed. The first, and from the scientific point of view undoubtedly the best, is the formal method of Weierstrass, Cantor, and others, which consists in defining numbers as symbols to which no concrete meaning is at first attached, but which can be combined with one another according to certain arbitrarily assigned laws. After the arithmetic of such numbers has been developed, it may be applied to various concrete (for instance, geometrical) questions.

The second method is to define numbers as symbols standing for certain concrete conceptions; for instance, the ordinary real numbers would be best defined as lengths measured upon a straight line. The obvious advantage of this method is the concreteness which it gives to the whole subject; an advantage so great that this appears to be the only practicable way of treating irrational numbers until the student has attained a very great degree of maturity. The plan which Mr. Hayward follows is the application of this method to the treatment of the imaginaries of ordinary algebra. The first subject taken up is therefore the geometrical one of *vectors*, the case in which the vectors lie in a single plane being considered almost exclusively. After devoting one chapter to the addition and subtraction of vectors, the multiplication of coplanar vectors is next taken up, the definitions here laid down being of course not those of quaternions. A particular vector is chosen as unit, its length being the unit of length and its direction being that from which the inclination of other vectors is measured. The product of two vectors is then defined as that vector which can be obtained from the multiplicand by the same amount of turning and stretching as is necessary to pass from the unit vector to the multiplier. The commutative, associative, and distributive laws are established, and the allied questions of division and raising to real powers are discussed. It is then shown—and here the question of complex numbers first comes in—that if we define the complex number $a + b\sqrt{-1}$ to be the vector whose components in the direction of the unit vector and in the direction obtained by turning this through the positive angle of 90° are a and b respectively, the complex quantities thus defined will obey all the ordinary laws of algebra. It will be seen that for Mr. Hayward complex numbers *are* vectors, whereas it seems much better to say merely that they *may be represented by* vectors. This may seem a small matter, but we are inclined to think that it is worth while to bring it clearly before the student at this point. *

* This is admirably done by De Morgan in his "Trigonometry and Double Algebra," 1849, a book which, as Mr. Hayward states in his preface, has served him in many respects as a model.

In chapter III comes the introduction of the trigonometric functions of a real angle, which are discussed graphically with some fulness. Many of the properties of these functions (addition theorems, etc.) are deduced in the following chapter directly from De Moivre's theorem. This method of obtaining these formulæ is not only extremely elegant, but has the advantage of avoiding, without any effort on the part of the reader, the necessity of considering separately the various cases which arise according as the angles lie in one or another quadrant.

"Vector Indices and Logarithms" is the title of chapter V, one of the least satisfactory chapters of the book. For instance, on page 108 the value of a certain constant (η) is determined on the assumption that it exists—an assumption which is nowhere justified. Later in the chapter we are pleasantly surprised by finding a rigorous, though perhaps not the best, treatment of the exponential limit.

Next follows a chapter on the hyperbolic functions, which is marred, as is in fact the whole book, by the use of new terms in place of universally accepted ones. The author "hopes to carry mathematicians with him in giving the name of *excircle* to the *rectangular hyperbola*." The hyperbolic functions are of course spoken of as excircular functions, though the ordinary notation for them is not changed. Other innovations in terminology are *project* and *traject* for the real and imaginary parts of a complex number respectively; and *logometer* (introduced by De Morgan) for the general logarithm.

A readable chapter on the roots of unity made clear by a constant reference to figures closes what may be regarded as the first part of the book.

This is followed by four chapters on infinite series and products. The first of these begins with an instructive and somewhat detailed graphical discussion of the geometric series. Then comes a rather brief treatment of the convergency of series of complex terms in general, including a treatment of the question of infinitely slow convergency. It goes without saying that we welcome the recent appearance of this subject in any form in English books. It is, however, an extremely difficult subject for the student to grasp, and should be amply illustrated by the graphical method for series with real terms before those with complex terms are taken up. Moreover, it is a subject in which the statements must be made with the greatest care if they are to convey the correct idea. Would it not have been better if Mr. Hayward had passed beyond the standpoint of Stokes and Seidel (now nearly fifty years old) and treated the subject, as it is now treated on the Continent, from the point of view of uniform

convergency, which is both more general and less likely to suggest false ideas?

In the following chapters—Expansions and Summations; Series of Products (i.e., infinite products); Series of Partial Fractions—many well-known matters are taken up and are often illustrated in an interesting graphical manner. As in the case of the exponential limit so again on page 279 we are surprised at finding a sound treatment of the development of $\sin u$ in an infinite product. Our surprise is, however, lessened when we note from the remark near the bottom of page 278 that Mr. Hayward does not appreciate the proof which he gives, that the product converges to the *right* value, but thinks it would be sufficient merely to prove that it converges.

The last chapter, on Rational and Integral Functions, is a slight introduction (geometrical, of course, as we believe such an introduction should be) to what is ordinarily spoken of as the theory of functions. A well-known but incomplete proof that every equation has a root will be found here which deserves special mention as it has crept into even so good a book as Harkness and Morley's Theory of Functions. Then follows Cauchy's theorem for determining the number of roots within a given contour; the proof given being, as Mr. Hayward says, simpler than that usually given. It must, however, be said that the possibility of factoring a rational integral function into linear factors has here been otherwise proved, while it is ordinarily desired to prove this fact as a corollary to Cauchy's theorem. Finally, the question of conformal representation is taken up, especially for cubic functions with real coefficients; but only the images of the lines parallel to the axes of reals and of imaginaries in the plane of the independent variable are considered. Among other things an interesting general theorem, due, we believe, to F. Lucas, concerning the asymptotes of these curves is given.

As has already been sufficiently emphasized, Mr. Hayward does not treat his subject with that degree of mathematical rigor which it seems to us to demand. On the other hand, the constant use of geometrical illustration gives the book a freshness and interest which many a mathematical writer might well envy.

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APOLAR TRIANGLES ON A CONIC.

BY PROFESSOR F. MORLEY.

§ 1. *Apolar Triads.*

Take two triangles, or point-triads, in a plane, say T and T', where (attaching complex numbers to the points in the usual way) T is t_1, t_2, t_3, and T' is t_1', t_2', t_3'. Take the polar pair of t_1' as to T, and the polar point of t_2' as to this polar pair, and let this polar point be t_3''. The relation thus imposed on T and T' is symmetric both as to the points T, the points T', and the two triads T and T'; it is, in fact,

$$s_1 s_2' - s_2 s_1' + 3(s_3 - s_3') = 0, \qquad (1)$$

where $s_1 = \Sigma t_\lambda$, $s_2 = \Sigma t_\mu t_\nu$, $s_3 = t_1 t_2 t_3$, and similarly for T'. Compare Salmon, Higher Algebra, § 151. Or in the symbolic notation, if T and T' are given by $a_t^3 = 0$, $a'_{t'}{}^3 = 0$, the relation is $(aa')^3 = 0$. The triads are now said to be *apolar*.

This covariant notion of apolarity, stated above for triads, is the natural extension of the notion of harmonic pairs, and can be immediately generalized, as is well known.

If the points lie on a line, we can deal with them projectively. Joining them to any point we have two apolar triads of a pencil. Pass a conic through the vertex of the pencil; the triads of rays cut out apolar point-triads of the conic. Express the co-ordinates of any point of the conic parametrically, say by

$$x : y : z = t^2 : t : 1;$$

then the parameters of the two triads of the conic obey the relation (1), inasmuch as the parameter t is proportional to the ratio in which a side of the triangle of reference is divided by the line joining the opposite vertex of the triangle to the point t of the conic. And in general this is sufficient justification for the interpretation of binary forms and their covariants by means of points of a conic.

Now if in the treatment of covariants by means of complex numbers (which I will call for shortness the inversive method) we restrict our view to points of a circle, any interpretation of a covariant (or rather of its vanishing), so obtained, can be at once stated also projectively for a conic. For if we take the circle as having the equation

$$xz = 1,$$

where x, z are conjugate complex numbers, this equation is a special form of the equation

$$xz = y^2$$

which we selected for the conic. In the transition from the inversive construction to the projective one, pairs of points inverse as to the circle must be replaced, for projective purposes, by the imaginary points where their line of symmetry meets the circle. In particular the centre of the circle and the point ∞ must be replaced by the circular points.

This identity of the results obtained so long as we restrict ourselves on the one hand to points on a circle and the pairs of inverse points, on the other to the single conic and auxiliary lines, is illustrated as far as concerns the cubic by comparing Beltrami's constructions for the covariants (for which I may refer to my article On the covariant Geometry of the Triangle, *Quar. Jour.*, vol. 25) with Salmon's Conics, note, p. 387 of the sixth edition. The identity is evident so far as it goes; but there is not in Salmon any projective construction for the polar pair of a given point. By drawing the canonical figure, in which the conic is a circle, the cubic is represented by an equilateral triangle, and the Hessian points are therefore the circular points, we can verify at once the following statement:—

Let t_1, t_2, t_3 be the fundamental points, j_1, j_2, j_3 the Jacobian points, x any other point of the conic. Draw xy_λ harmonic with $t_\lambda j_\lambda$; then the lines $t_\lambda y_\lambda$ meet at the pole of the line required. This pole lies on the Hessian line, so that the line required passes through the intersection of the three lines $\overline{t_\lambda j_\lambda}$.

The inversive point of view is taken in the special problem of the next section; the rest of the article is projective, but the statements are made for the point as primary element, and are not repeated in the reciprocal form. And in accordance with this one-sided mode of statement, the word triangle is often loosely used here as meaning point-triad.

§ 2. *Feuerbach's Theorem.*

Many of the proofs which have been given of this remarkable and familiar theorem employ the method of ordinary inversion; but, so far as I know, none of them have stated the covariant aspect of the matter. The theorem is, for my purpose, as follows:

From a triangle T, and an auxiliary point x, form a new triangle T_x by taking the harmonic of x as to each pair of points of T. From T_x and an auxiliary point y form in the same way a third triangle T_{xy}. Let x and y be inverse points as to the circumcircle (T) of T. Then the circles (T) and (T_{xy}) touch at a point t.

To reduce this statement to the usual form, take for x the centre of (T), and therefore for y the point ∞. Then T_x is

a triangle whose sides touch (T'), and the middle points of these sides make up T_{xy}.

It is natural to suppose that *the points x, y, t are apolar with T*; this I have to verify.

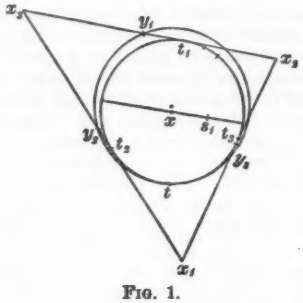

FIG. 1.

Taking $x = 0, y = \infty$, we have from (1)

$$t = s_2/s_1. \tag{2}$$

The harmonic of 0 as to t_μ and t_ν is

$$x_\lambda = 2t_\mu t_\nu/(t_\mu + t_\nu).$$

The harmonic of ∞ as to x_μ and x_ν is

$$\begin{aligned} y_\lambda &= \tfrac{1}{2}(x_\mu + x_\nu) \\ &= t_\lambda\left(\frac{t_\nu}{t_\lambda + t_\nu} + \frac{t_\mu}{t_\lambda + t_\mu}\right). \end{aligned}$$

Hence

$$y_\lambda - t_\lambda = t_\lambda \cdot \frac{t_\mu t_\nu - t_\lambda^2}{(t_\lambda + t_\mu)(t_\lambda + t_\nu)},$$

$$\begin{aligned} t - y_\lambda &= \frac{t_\mu t_\nu}{s_1}\left(1 - \frac{t_\lambda}{t_\lambda + t_\nu} - \frac{t_\lambda}{t_\lambda + t_\mu}\right) \\ &= \frac{t_\mu t_\nu}{s_1}\,\frac{t_\mu t_\nu - t_\lambda^2}{(t_\lambda + t_\mu)(t_\lambda + t_\nu)}, \end{aligned}$$

and therefore

$$\frac{y_\lambda - t_\lambda}{t - y_\lambda} = \frac{s_1 t_\lambda}{t_\mu t_\nu}. \tag{3}$$

Now introduce the condition that the origin is the centre of the circle (T); supposing this circle to have the radius 1, then

$$\left|\frac{y_\lambda - t_\lambda}{t - y_\lambda}\right| = \delta,$$

where $\delta = |s_1|$ = distance between the circumcentre and orthocentre of T.

Now if the stroke from t to y_λ meet (T) again at z_λ, we have, by elementary geometry (Euclid iii. 36),

$$\left|\frac{y_\lambda - t_\lambda}{t - y_\lambda}\right| = \left|\frac{z_\lambda - y_\lambda}{y_\lambda - t_\lambda}\right|.$$

Hence

$$\left|\frac{z_\lambda - y_\lambda}{t - y_\lambda}\right| = \delta^2,$$

that is, z_λ divides the stroke from t to y_λ in the ratio $1 - \delta^2 : \delta^2$. Since this ratio is independent of λ, the circles (T_{xy}) and (T) touch at t.

We have taken as origin the circumcentre. If also we take as real axis the line through the centroid, so that now s_1 is real and $= \delta$, then

$$1/t_1 + 1/t_2 + 1/t_3 = \delta,$$

and

$$s_3 = \delta s_3.$$

Therefore

$$t_3 = s_3/\delta, = s_3.$$

Thus the triangle T is given by the equation

$$t_\lambda^3 - t - \delta(t_\lambda^2 - t_\lambda t) = 0, \qquad (4)$$

and this for different values of δ represents an involution of triangles on a circle, any two of which are apolar, as appears on applying the condition (1). But it includes also triangles of which one point is on the circle, and the other two are inverse as to the circle, for when we write for t_λ and t their reciprocals, we get the same equation, and we can suppose either that t_λ is conjugate with $1/t_\lambda$, when $\lambda = 1, 2, 3$, or that t_μ is conjugate with $1/t_\nu$, in which case one pair of points is an inverse pair.

For three points on the circle we have, from $t = s_3$, the theorem that *the orientation of the radii to the three points is constant and equal to that of the radius to t*, the orientation being the sum of the angles made with the selected real axis.

§ 3. *Doubly Apolar Conics.*

An involution of triads of which each pair is apolar, such as was given by (4), may be called an *apolar involution.* The simplest view of it is that it consists of the polar triads of a self-apolar* tetrad. For, first, if we take the equations of any two triads, as

$$(a, b, c, d)(x, y)^3, \quad (b, c, d, e)(x, y)^3,$$

as in Salmon's Higher Algebra, § 203, then when the triads are apolar

$$ae - 4bd + 3c^2 = 0,$$

and the tetrad of which they are polar triads is self-apolar. And second, taking for the equation of the tetrad

$$x^4 + 4xy^3 = 0,$$

it is verified at once that any triad which is apolar with two polar triads is itself a polar triad.

Such an involution will now be considered in connexion with a special system of conics.

Inversively, we say that an equilateral triangle and the point ∞ are self-apolar. Hence projectively we can say that adjacent corners of a regular hexagon in a circle, and the circular points, are self-apolar points of the circle.

Now imagine a quilt formed of equal regular hexagons, say of side 1; draw circles round the hexagons, and take two visibly intersecting circles. Their four intersections are self-apolar points of each circle.

Referring to Salmon's Conics, p. 336, Ex. 3, we see that the intermediate invariants Θ and Θ' both vanish; for we have

$$r = r' = 1, \quad \alpha^2 + \beta^2 = 3.$$

Reye calls a conic in point co-ordinates and a conic in line co-ordinates apolar when the bilinear invariant—Θ or Θ', as the case may be—vanishes; hence our two circles are sufficiently characterized if we may say that they are doubly apolar.

Hence through four points—two real and two imaginary—two doubly apolar conics can be drawn; for these conics $\Theta = \Theta' = 0$. And it is easy to prove, conversely, that when $\Theta = \Theta' = 0$, the intersections of the conics are a self-apolar system on each. Cf. Clebsch-Lindemann-Benoist, vol. I. p. 375.

* The term *self-apolar* is used in preference to the clumsy word equianharmonic, which is, moreover, sometimes used with reference to any four points on a line and their projection.

§ 4. *Wolstenholme's Configuration.*

With two such conics (T) and (U), we can associate a third conic (V), which is equally the reciprocal of (T) as to (U), that of (U) as to (T), the envelope of lines divided harmonically by (T) and (U), and the locus of points divided harmonically by (T) and (U). Moreover, the relations of the three are entirely symmetrical. This system of three conics, of which each pair is doubly apolar, was studied by Wolstenholme, Problems, pp. 263–4; but his results are stated without reference to the theory of binary forms, and it is desirable to indicate the connexion.

The quilt arrangement is not so convenient, for the system of three conics, as the arrangement to which Wolstenholme's canonical equations lead. These equations are

$$\begin{array}{ccccc} x^2 + 2yz = 0, & & y^2 + 2zx = 0, & & z^2 + 2xy = 0, \\ \text{for} \quad (T), & & (U), & & (V), \text{ respectively.} \end{array}$$

Taking the fundamental triangle equilateral and the co-ordinates areal (Fig. 2)*, these give three rectangular hyperbolas; the real intersections form a regular hexagon, and the real foci form an equal regular hexagon.

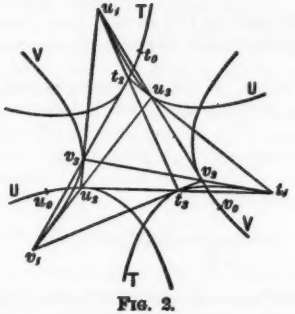

FIG. 2.

Selecting (T) as the conic on which apolar triangles are to be considered, we take for the co-ordinates of any point t on it,

$$x : y : z = t : -t^2 : \tfrac{1}{2}.$$

The intersections of (T) and (U) are points on (T) given by

$$t^4 + t = 0; \qquad (5)$$

* How to do this appears from a remark of Clifford, Papers, p. 412.

these we take as the fundamental self-apolar tetrad on (T). Now taking any point t_0, its polar triad, t_λ, t_μ, t_ν, is given by

$$t_0(4t^3+1)+3t=0, \tag{6}$$

so that

$$\Sigma t_\lambda = 0, \quad \Sigma t_\mu t_\nu = 3/4t_0, \quad t_\lambda t_\mu t_\nu = -1/4. \tag{7}$$

Rotating the figure round its centre through angles $2\pi/3$, the points t_α pass into the points u_α and v_α, where $\alpha = 0, 1, 2, 3$; the co-ordinates of all these points being given by the scheme:

$$\begin{array}{l} t_\alpha \text{ is } t_\alpha, \ -t_\alpha^2, \ \tfrac{1}{2}, \\ u_\alpha \text{ is } \tfrac{1}{2}, \ t_\alpha, \ -t_\alpha^2, \\ v_\alpha \text{ is } -t_\alpha^2, \ \tfrac{1}{2}, \ t_\alpha. \end{array}$$

Now the line joining t_μ and t_ν is

$$x(t_\mu + t_\nu) + y - 2zt_\mu t_\nu = 0,$$

or from (7)

$$-2xt_\lambda^2 + 2yt_\lambda + z = 0,$$

and this is the tangent of (U) at u_λ.

So also the sides of the triangle u_1, u_2, u_3 touch (V) at v_1, v_2, v_3.

The triangle u_1, u_2, u_3 is harmonic as to (T). For two points x_1, y_1, z_1, and x_2, y_2, z_2 are harmonic as to $x^2+2yz=0$ when

$$x_1x_2 + y_1z_2 + y_2z_1 = 0;$$

and in this case the left side

$$\begin{array}{l} = \tfrac{1}{4} - t_\mu t_\nu (t_\mu + t_\nu) \\ = 0. \end{array}$$

Thus Wolstenholme's main results are verified.

Next, *the three lines* $\overline{t_\lambda u_\lambda}$ *meet at the point* t_0, *with which we began.*

For the join of t_0 and t_λ is

$$x(t_0 + t_\lambda) + y - 2zt_0t_\lambda = 0;$$

this passes through u_λ if

$$\tfrac{1}{2}(t_0 + t_\lambda) + t_\lambda + 2t_0t_\lambda^2 = 0,$$

which is equation (6).

Similarly, the join of t_0 and u_0 passes through the polar point of t_0.

The tangent at u_0 meets (T) at the (imaginary) Hessian points of t_1, t_2, t_3. For this tangent is

$$-2xt_0^2 + 2yt_0 + z = 0;$$

it meets (T) where

$$-2tt_0^2 - 2t_0t^2 + \tfrac{1}{2} = 0;$$

and the left is the Hessian of the left of (6).

The tangent at v_0 meets (T) at the polar pair of t_0. For this tangent is

$$x - 2yt_0^2 + 2zt_0 = 0;$$

it meets (T) where

$$t + t_0 + 2t^2t_0^2 = 0,$$

and the left is the second polar of t_0 as to the fundamental tetrad.

Lastly, *the lines $\overline{v_0t_\lambda}$ meet (T) again at the Jacobian of t_1, t_2, t_3.* For evidently $v_0, t_\lambda, v_\lambda$ are collinear; and v_λ is the pole of $\overline{t_\mu t_\nu}$ as to (T). Hence $\overline{v_0t_\lambda}$ and $\overline{t_\mu t_\nu}$ are harmonic as to (T); but harmonic lines meet a conic in harmonic pairs.

The configuration is determined when we take a conic and on it two apolar triangles. To return to the original case of a circle, and on it two triangles t_1, t_2, t_3, and $t, 0, \infty$, § 2, we project figure 2 so that the points where a tangent of (U) meets (T) pass into the circular points. (T) becomes the circle, (U) a parabola touching the sides of the system of triangles T, (V) a rectangular hyperbola on which lie all the points of the tangent triangles T'_0.

§ 5. *The Complementary Line.*

In general a line which cuts two plane curves of degree 3, say $a_x^3 = 0$, $b_y^3 = 0$, in two apolar triads, envelops a curve of class 3, in Clebsch's notation $(abu)^2 = 0$. Compare Clebsch's Geometry, vol. I. p. 344 *et seq.* of the French edition.* Let the given cubics reduce to line-triads, and let these triads touch a conic and be apolar triads of that conic; then the class-cubic is clearly this conic and some complementary point. Reciprocally, then, in our case, where we begin with two

* Clebsch gives the method; the fact is stated in some lecture notes of Clifford, Works, p. 534. It appears that the equation by which Clifford defines apolar (or harmonic) triads should have only three terms on the left, as the second three terms are only a rewriting of the first three.

apolar point-triads of a conic, the locus of points which are divided apolarly by the two triads is the conic itself and some complementary line. It remains to identify this line.

In the study of harmonic pairs the degenerate case when one of the elements is arbitrary is of great use. Then we know the other elements all coincide. Here, in the study of apolar triads, *the Hessian pair of a triad and an arbitrary element are apolar with that triad.* For we know that the polar pair of an arbitrary point, as to a point-triad, is harmonic with the Hessian pair.

Now, taking the triangles t_1, t_2, t_3 and t, 0, ∞, where t is Feuerbach's point (§ 2), we know, first, that lines forming an equilateral triangle determine on the line infinity a triad whose Hessian pair is the circular points 0, ∞, so that three such lines on the one hand and an isotropic pair and an arbitrary line on the other cut the line infinity apolarly; and we know, secondly, from elementary geometry (*Quar. Jour.*, vol. 25, p. 190) that there are two points e, the lines from which to t_1, t_2, t_3 form a vanishing equilateral triangle—those points, namely, which have been called the equiangular points of the triangle. Hence the line-triads from e to t_1, t_2, t_3 and t, 0, ∞ are apolar; for the pencil is cut apolarly by the line infinity. The equiangular points do not lie on the circumcircle, hence *any point on the join of the equiangular points is divided apolarly by the two triads.*

Thus the complementary line is determined for this case. To pass to a covariant statement, we notice (*Q. J., loc. cit.*) that the line passes through the symmedian point of $t_1t_2t_3$, that is, through the pole of the Hessian line. Thus given two apolar triads on a conic, the points divided apolarly by them lie either on the conic itself or on the line through the poles of their Hessian lines, that is, on *the line which meets the conic at the Jacobian of the Hessians of the triads.*

In conclusion, it is hoped that the instance of Apolarity which has now been worked out may be useful to the student of Meyer's work, *Apolarität und Rationale Curven*, to which, above all, reference must be made for the projective development of the theory.

HAVERFORD COLLEGE.

AN INSTANCE WHERE A WELL-KNOWN TEST TO PROVE THE SIMPLICITY OF A SIMPLE GROUP IS INSUFFICIENT.

IN the December number of this journal (page 64, footnote) Professor Moore asks whether an instance is known where the test used by Klein in his "Vorlesungen über das

Ikosaeder" (1884), page 18, to prove that the icosahedron-group of rotations is simple does not apply. The alternating substitution group of degree 68 is such an instance. The number of substitutions of the form $ab'c$ is

$$\frac{68 \cdot 67 \cdot 66}{3} = 100232,$$

and

$$1 + 100232 = 9 \cdot 7 \cdot 37 \cdot 43.$$

The diophantine equation

$$1 + 100232\alpha + \ldots = d$$

has in this case at least the following three solutions:

$$(\alpha, \beta, \delta, \ldots; d) = (1, 0, 0, \ldots; 1),\ (1, 1, 1, \ldots; \frac{68!}{2}),$$
$$(1, 1, 0, \ldots; 100233).$$

Since every alternating group whose degree exceeds four is simply isomorphic to a number of other simple groups, this instance proves that the given test is insufficient with respect to simple groups which are not alternating. The subgroup Γ_n of Professor Moore's article and its constituent groups are clearly such simple groups, if we take for G_n the given alternating group.

GEORGE A. MILLER.

December 28, 1894.

BRIEFER NOTICES.

LOBACHEVSKY MEMORIAL VOLUME: 1793–1893. *Celebration of the one hundredth anniversary of the birth of N. I. Lobachevsky.* [In Russian.] Kazàn, University Press, 1894. Folio, 212 pp. With a portrait of Lobachevsky.

IN addition to a detailed account of the three days' celebration in honor of Lobachevsky, held at the University of Kazàn in November, 1893,* this volume contains the letters and telegrams of congratulation received by the university, and some of the addresses and papers read on this occasion. Professor Savòrov gives a somewhat popular exposition of the meaning of non-Euclidean geometry, while Professor Smirnòv discusses the same subject rather elaborately from the philosophical point of view. The other papers are historical: Mr. Iznòskov speaks of Lobachevsky's activity as a member of the Kazàn Agricultural Society; the president of the university,

* See *Bulletin of the New York Mathematical Society*, vol. 3, p. 201.

Professor Voroshilov, devotes his inaugural address to a sketch of the remarkable pedagogic and administrative activity of Lobachevsky, who for nineteen years (1827 to 1846) held the office of president of the university; Professor Vasiliev's paper, the most interesting of all, is a succinct account of Lobachevsky's life. An English translation of this biography was recently published by Professor G. B. Halsted, of the University of Texas. A. Z.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held Saturday afternoon, January 26, at three o'clock, the president, Dr. Hill, in the chair. There were ten members present. On the recommendation of the council, Professor Irving Fisher, of Yale University, New Haven, and Mr. William A. Freedman, of Columbia College, New York, nominated at the preceding meeting, were elected to membership. Six nominations for membership were received. The following papers were presented:

(1) "The principles of differentiation in space-analysis," by Professor A. MACFARLANE.

(2) "Apolar triangles on a conic," by Professor F. MORLEY.

(3) "The group of automorphic transformations of a bilinear form," by Professor H. TABER.

PROFESSOR CAYLEY died at Cambridge, England, on January 26, at the age of seventy-four years.

ON the 17th of December the annual award of prizes took place at the French Academy. Dr. JULIUS WEINGARTEN, professor of mechanics in the Charlottenburg Polytechnic School, received the grand prix de mathématiques, honorable mention being accorded to C. GUICHARD, professor of mechanics in the Faculty of Sciences at Clermont-Ferrand. The subject set by the Academy for this prize was: to advance in some important respect the theory of the deformation of surfaces.

The Bordin prize (to study the problems of theoretical mechanics admitting integrals algebraic in the velocities, especially quadratic integrals) was awarded to PAUL PAINLEVÉ of the Sorbonne, LIOUVILLE and ELLIOT receiving honorable mention.

The Francœur prize was given to J. COLLET; the Poncelet prize to H. LAURENT, for the whole of his mathematical works; the Dalmont prize to AUTONNE and MAURICE D'OCAGNE, with honorable mention to POCHET and WILLOTTE.

Professor GEORGE A. HALE, of Chicago, was awarded the Janssen gold medal for his work in physical astronomy; Admiral JAVELLE received the Lalande prize for his observations of nebulæ; M. CONIEL, the Valz prize for various astronomical labors. The Damoiseau prize (to perfect the methods for computing the perturbations of the smaller planets, etc.) was given to Dr. M. BRENDEL, of Greifswald.

The subjects set by the French Academy for the next mathematical prizes are as follows: *grand prix* (Oct. 1, 1895), to advance in some important respect the algebraic theory of groups of substitutions of n letters; *prix Bordin* (Oct. 1, 1895), to advance in some important respect the theory of geodesic lines, the case of a linear element with any number of variables not being excluded by the Academy; *prix Damoiseau* (June 1, 1896), to connect, by means of the theory of perturbations, the different apparitions of Halley's comet, going back to Toscanelli's observations in 1456 and taking into account the attraction of Neptune, and to calculate with precision the next return of this comet in 1910; *prix Damoiseau* (June 1, 1898), to give an exposition of the theory of the perturbations of Saturn's satellite Hyperion, discovered in 1848 simultaneously by Bond and Lassell, taking into account principally the action of Titan, to compare the observations with the theory, and to deduce therefrom the value of the mass of Titan.

THE meeting of the American Association for the Advancement of Science will be held this summer at Springfield, Mass. The general sessions will begin on Thursday morning, August 29, at ten o'clock.

THE University of Göttingen has called Professor D. Hilbert from Königsberg to succeed to the professorship of mathematics vacated by Professor H. Weber, who was recently called to Strassburg.

A graduate student at Clark University, Mr. H. G. Keppel, is taking a series of photographs of the mathematical models and portraits of mathematicians to which he has access. It will include stereoscopic views of about one hundred different models. Among the portraits already photographed are likenesses of Sophie Kovalevsky, Gauss, Abel, Euler, Newton, Laplace, Lagrange, John Bernoulli, Borchardt, Kronecker, Cayley, Benjamin Peirce, Kummer, Steiner, Monge, Lobachevsky, Bessel, Abbé Moigno, Möbius, Fourier, Fermat, Jacobi, De Morgan, Galileo, Huygens, Lambert, Dirichlet, Pascal, Galois, Montucla, Lalande, Kepler, Arago, Tycho Brahe. Applications for copies of Mr. Keppel's pictures may be addressed to him and will receive his attention.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APPELL (P.) et GOURSAT (E.). Théorie des fonctions algébriques et de leurs intégrales. Étude des fonctions analytiques sur une surface de Riemann. Paris, Gauthier-Villars, 1894. 8vo. (10 and 256 pp. published.) Fr. 14.00

BELLACCHI (G.). Introduzione storica alla teoria delle funzioni ellittiche. Firenze, Barbèra, 1894. 8vo. 4 and 316 pp. Illustrated. Fr. 6.00

BULLETIN des mathématiques spéciales, publié sous la direction de B. Niewenglowski. Première année. Paris. 8vo.

CRANZ (H.). Lehrbuch der analytischen Geometrie der Ebene. 2ter Teil: Die einzelnen Linien zweiten Grades. Bearbeitet nach System Kleyer. Stuttgart, Maier, 1894. 8vo. 7 and 428 pp. Mk. 8.00

CWOJDZIŃSKI (T.). Anwendung der Fuchs'schen Theorie auf die Differentialgleichung der Gauss'schen hypergeometrischen Reihe. Brody, West, 1894. 8vo. 45 pp. Mk. 1.00

DIXON (A. C.). The elementary properties of the elliptic functions; with examples. London and New York, Macmillan, 1894. 8vo. 8 and 142 pp. Cloth. $1.25

FROBENIUS (G.). Gedächtnissrede auf Leopold Kronecker. Berlin, Reimer, 1893. 4to. Mk. 1.50

JAHRESBERICHT der Deutschen Mathematiker-Vereinigung. Vol. III (1892–93). Herausgegeben von W. Dyck und E. Lampe. Berlin, Reimer, 1894. 8vo. 4 and 602 pp. Illustrated. Mk. 16.00

MANNING (E. P.). On the representation of a function by a trigonometric series. Dissertation presented to the board of university studies of the Johns Hopkins University for the degree of Doctor of Philosophy. Baltimore, Friedenwald, 1894. 4to. 6 and 27 pp.

NAETSCH (F. W. E.). Zur Theorie der homogenen linearen Differentialgleichungen mit doppelt-periodischen Coefficienten. [Diss.] Leipzig, 1894. 8vo. 60 pp.

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte I: Calcolo differenziale. 10 and 316 pp. Parte II. Calcolo integrale. 6 and 318 pp. Milano, Hoepli, 1895. 16mo. Illustrated.

PEANO (G.). Estensione di alcune teoremi di Cauchy sui limiti; nota. Torino, Clausen, 1894. 8vo. 24 pp. [*Atti della R. Accademia delle scienze di Torino*, Vol. 30.]

SÄRCHINGER (E.). Beitrag zur Theorie der Funktionen des elliptischen Cylinders. [Diss.] Leipzig, 1894. 4to. 28 pp.

VUKIČEVIČ (P. L.). Die Invarianten der linearen homogenen Differential-Gleichungen nter Ordnung. [Diss.] Berlin, 1894. 4to. 42 pp.

ZERMELO (E.). Untersuchungen zur Variations-Rechnung. [Diss.] Berlin, Mayer & Müller, 1894. 8vo. 97 pp. Illustrated. Mk. 2.50

II. ELEMENTARY MATHEMATICS.

FABRIS (V.). Nozioni elementari di algebra, ad uso della terza classe delle scuole tecniche, secondo i programmi governativi 8 nov. 1888, 31 ag. 1892. 3a edizione. Padova, Sacchetto, 1895. 16mo. 41 pp. Fr. 0.80

FACIOLÀ (O.). Complementi di geometria: Le proprietà del triangolo e parallele sulla indipendenza dal loro postulato; ad uso delle scuole secondarie. Parte I. Messina, d'Angelo, 1894. 8vo. 76 pp. Fr. 1.00

FUCINI (C.). Algebra elementare. Parte I. Genova, Colombo, 1894. 16mo. 199 pp. Fr. 2.00

FULCHERIS (P.). Elementi di geometria, ad uso delle scuole tecniche e normali. 15ma edizione, ordinata secondo i programmi 21 giugno 1885. Torino, Paravia, 1895. 16mo. 122 pp. 14 plates. Fr. 2.00

GROISSL (J.). Die Absolutorial-Aufgaben aus der Mathematik und Physik an den humanistischen Gymnasien Bayerns. Theil I: Anleitungen zur Lösung und Resultate. Theil II: Die Aufgaben von 1889 bis 1894. Zur Selbstübung und Vorbereitung für das Absolutorium bearbeitet. München, Zipperer, 1894. 8vo. 4, 19, and 69 pp. Mk. 2.00

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